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Solving the problem of two viscous incompressible fluid media in the case of constant phase saturations

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Abstract: The solution to equations of two viscous homogeneous incompressible fluid media with the pressure phase equilibrium in the case of a constant phase is obtained. The influence of the physical phase densities, saturation, volume and viscosity of substances constituting a two-phase continuum in the flow velocity and pressure is shown. Also, the solution admitting a limiting transition to the known solution of the problem of a flow of a viscous incompressible single-phase medium is constructed.

Keywords: mathematical modeling; two-velocity hydrodynamics; incompressible fluid; Earth's tectonosphere

1 Introduction

Recently, in various fields of science and technology, considerable attention of scientists has been given to the study of nonlinear dynamic systems. The study of physical and technical processes in the continuum mechanics begins with constructing of a mathematical model. In this case, dynamic systems to be obtained are as a rule, nonlinear. Their investigation has resulted in the discovery of such phenomena as deterministic chaotic behavior, principal unpredictability of events and self-organization. Geology deals with studying the Earth as a dynamic system. The vast majority of interactions in natural geological conditions are irreversible nonlinear processes: the response is always proportional to the action. Geology deals with the results of interactions such as displacements, strains, and formation of various structures of different sizes. Most geological processes occurring in the Earth's interior are confined to its external lateral heterogeneity in the structure and composition of the spherical shell thickness of ~ 700 km, called tectonosphere. The motion and deformation observed in tectonosphere are on the one hand due to the physical and chemical processes and gravitational differentiation and on the other hand - due to the trend of the tectonosphere isostatic equalization (compensation). Analysis of a number of geophysical data [1] has allowed one to assume that in conditions of progress of geological processes ($t \sim 10^3 \dots 10^7$ years) and large geological bodies ($L \sim 10 \dots 10^6$ m), the Earth's tectonosphere can be considered to be a highly viscous incompressible multiphase fluid medium. Therefore, a continuous medium at the geological time scale can be represented as a viscous "fluid-1" due to intrinsic viscosity, for other reasons, it attacks the necessary thermodynamic conditions of the phase transition. Along the grain boundaries and the inter-grains there begins accumulation of magma, *i.e.* "fluid-2" with a viscosity intrinsic of melts known in geology. This melt is included in the process of the combined heat and mass transfer and is filtered through the system that produced it. In other words, this model represents the dynamics of the mutual penetration of one fluid that is less viscous through a medium with greater viscosity as a kind of filtering process. Or, by analogy with a one-speed system of the Navier-Stokes equations, this model can be called a two-speed system of the Navier-Stokes equations or a two-velocity hydrodynamics. In these circumstances, the tectonosphere movement is dissipative and may be described by a two-speed system of the Navier-Stokes equations with the pressure phase equilibrium [2–4], which, in these conditions, becomes much simpler and has the following form:

$$\nu_1 \Delta \mathbf{u}_1 = \nabla p - \rho \mathbf{f}, \quad \mathrm{div}\mathbf{u}_1 = 0, \tag{1}$$

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$$\nu_2 \Delta \mathbf{u}_2 = \nabla p - \rho \mathbf{f}, \quad \mathrm{div}\mathbf{u}_2 = 0, \tag{2}$$

In formulas (1) and (2) $\mathbf{f} = (f_1, f_2, f_3)$ is the mass force per unit mass, $\mathbf{u}_i = (u_{1i}, u_{2i}, u_{3i})$ are the velocities of the i-th phase, $i = 1, 2$, p is the pressure, Δ and ∇ are the Laplace operator and the gradient of $\mathbf{x} = (x_1, x_2, x_3)$, respectively, $\rho = \rho_1 + \rho_2$ is the common density of the two-velocity continuum, $\nu_1(\rho_1)$ and $\nu_2(\rho_2)$ are the corresponding shear viscosities (partial densities) of the phases [2, 5].

This work is organized as follows. In section 2 we give formulation of the problem to the equations of two-fluid viscous fluids. Section 3 we construct a solution to the equations of two-fluid viscous fluids, which in some cases allows one to obtain the analytical form of solution to the problem of more complex models of different areas of the Earth's tectonosphere.

2 Formulation of the problem

It is required to find $\mathbf{u}_i(\mathbf{x})$ $(i = 1, 2)$ and $p(\mathbf{x})$ is the solution of (1), (2) for some arbitrary bounded domain $\Omega \subset R^3$ which is 3D Euclidean space. Let us assume that the required functions $\mathbf{u}_i(\mathbf{x})$ $(i = 1, 2)$ and $p(\mathbf{x})$ satisfy the conditions at infinity [6]:

$$\mathbf{u}_i(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad p(\mathbf{x}) = O\left(|\mathbf{x}|^{-2}\right), \quad \text{for} \quad |\mathbf{x}| \to \infty \quad (i = 1, 2), \tag{3}$$

where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Assume that the function $\mathbf{f}(\mathbf{x})$ is continuous together with its first derivatives in a bounded domain $\tilde{\Omega} \subset \Omega$ and has a finite support $supp\mathbf{f}(\mathbf{x}) = \tilde{\Omega}$. In addition, we assume that the domains $\tilde{\Omega}$ and Ω are bounded by the piecewise smooth surfaces $\partial\tilde{\Omega}$ and $\partial\Omega$.

3 The solution of the problem

Let us apply the divergence operator to both sides of equation (1) and with allowance for second equation in (1), we obtain the Poisson equation for the pressure

$$\Delta p = \rho \, \mathrm{div}\mathbf{f}. \tag{4}$$

This equation can be obtained from system (2). Note that system (1), (2) is overdetermined [5].

Theorem. To solve problem (1)–(4) we have the following formulas:

$$\mathbf{u}_1(\mathbf{x}) = \frac{\rho}{4\pi\nu_1} \int_\Omega \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_\varsigma + \frac{\rho}{8\pi\nu_1} \nabla \int_\Omega (\nabla R(\mathbf{x}, \boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma})) d\Omega_\varsigma,$$

$$\mathbf{u}_2(\mathbf{x}) = \frac{\rho}{4\pi\nu_2} \int_\Omega \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_\varsigma + \frac{\rho}{8\pi\nu_2} \nabla \int_\Omega (\nabla R(\mathbf{x}, \boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma})) d\Omega_\varsigma,$$

$$p(\mathbf{x}) = -\frac{\rho}{8\pi} \Delta \int_\Omega R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_\varsigma.$$

Proof. Since the function div$\mathbf{f}$ is equal to zero outside the domain Ω due to the finiteness of the function $\mathbf{f}(\mathbf{x})$, and the domain Ω is bounded by the piecewise smooth surface $\partial\Omega$, the solution of equation (4), satisfying conditions (3) at $|\mathbf{x}| \to \infty$, can be written down as

$$p(\mathbf{x}) = -\frac{\rho}{4\pi} \int_\Omega \frac{\mathrm{div}\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_\varsigma. \tag{5}$$

In formula (5), $R(\mathbf{x}, \boldsymbol{\varsigma}) = |\mathbf{x} - \boldsymbol{\varsigma}|$ is the distance between the points $\mathbf{x}$ and $\boldsymbol{\varsigma}$.

Since the function $R(\mathbf{x}, \boldsymbol{\varsigma})$ is a function of difference, we can transform formula (5) using the well-known formula $\Delta R(\mathbf{x}, \boldsymbol{\varsigma}) = \frac{2}{R(\mathbf{x}, \boldsymbol{\varsigma})}$ and obtain

$$p(\mathbf{x}) = -\frac{\rho}{8\pi} \Delta \int_\Omega R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_\varsigma. \tag{6}$$

Excluding the pressures from the first equations in (1), (2) with allowance for (5), we obtain

$$\Delta \left[\nu_1 \mathbf{u}_1(\mathbf{x}) + \frac{\rho}{8\pi} \nabla \int_\Omega R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_\varsigma \right] = -\rho \mathbf{f}, \tag{7}$$

$$\Delta \left[\nu_2 \mathbf{u}_2(\mathbf{x}) + \frac{\rho}{8\pi} \nabla \int_\Omega R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_\varsigma \right] = -\rho \mathbf{f}. \tag{8}$$

The system of equations (7) and (8) is the Poisson equations system for the function, presented in the square brackets. Since the function $\mathbf{f}(\mathbf{x})$ is finite, and the unknown velocities $\mathbf{u}_i(\mathbf{x})$ $(i = 1, 2)$ satisfy conditions (3) at infinity, then the system of equations (7) and (8) can be written down as

$$\nu_1 \mathbf{u}_1(\mathbf{x}) + \frac{\rho}{8\pi} \nabla \int_\Omega R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_\varsigma = \frac{\rho}{4\pi} \int_\Omega \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_\varsigma,$$

$$\nu_2 \mathbf{u}_2(\mathbf{x}) + \frac{\rho}{8\pi} \nabla \int_\Omega R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_\varsigma$$

$$= \frac{\rho}{4\pi} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma}.$$

Hence, in order to determine the velocities $\mathbf{u}_i(\mathbf{x})$ $(i = 1, 2)$ on the one hand, we obtain the following expressions:

$$\mathbf{u}_1(\mathbf{x}) = \frac{\rho}{4\pi\nu_1} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} - \frac{\rho}{8\pi\nu_1} \nabla \int_{\Omega} R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_{\varsigma}, \tag{9}$$

$$\mathbf{u}_2(\mathbf{x}) = \frac{\rho}{4\pi\nu_2} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} - \frac{\rho}{8\pi\nu_2} \nabla \int_{\Omega} R(\mathbf{x}, \boldsymbol{\varsigma}) \mathrm{div}\mathbf{f}(\boldsymbol{\varsigma}) d\Omega_{\varsigma}, \tag{10}$$

and, on the other hand, between the velocities $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$ due to predetermination of system (1), (2) there is a relation [5]:

$$\nu_1 \mathbf{u}_1(\mathbf{x}) = \nu_2 \mathbf{u}_2(\mathbf{x}). \tag{11}$$

From (9), (10) taking into account the Gauss-Ostrogradsky formula, we obtain the following expressions:

$$\mathbf{u}_1(\mathbf{x}) = \frac{\rho}{4\pi\nu_1} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_1} \nabla \int_{\Omega} (\nabla R(\mathbf{x}, \boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma})) d\Omega_{\varsigma} - \frac{\rho}{8\pi\nu_1} \nabla \int_{\partial\Omega} (n(\boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma}))\, R(\mathbf{x}, \boldsymbol{\varsigma}) dS_{\varsigma}, \tag{12}$$

$$\mathbf{u}_2(\mathbf{x}) = \frac{\rho}{4\pi\nu_2} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_2} \nabla \int_{\Omega} (\nabla R(\mathbf{x}, \boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma})) d\Omega_{\varsigma} - \frac{\rho}{8\pi\nu_2} \nabla \int_{\partial\Omega} (n(\boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma}))\, R(\mathbf{x}, \boldsymbol{\varsigma}) dS_{\varsigma}, \tag{13}$$

where $(\mathbf{a}, \mathbf{b})$ is the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$, $n(\boldsymbol{\varsigma})$ is the unit vector of the external normal to $\partial\Omega$ at the point $\boldsymbol{\varsigma}$, dS_{ς} is the surface element $\partial\Omega$, containing the point $\boldsymbol{\varsigma}$.

By virtue of the finiteness of the function $\mathbf{f}(\mathbf{x})$ the surface integrals are zero, the expressions for the velocities $\mathbf{u}_i(\mathbf{x})$ $(i = 1, 2)$ acquire the following final form:

$$\mathbf{u}_1(\mathbf{x}) = \frac{\rho}{4\pi\nu_1} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_1} \nabla \int_{\Omega} (\nabla R(\mathbf{x}, \boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma})) d\Omega_{\varsigma}, \tag{14}$$

$$\mathbf{u}_2(\mathbf{x}) = \frac{\rho}{4\pi\nu_2} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_2} \nabla \int_{\Omega} (\nabla R(\mathbf{x}, \boldsymbol{\varsigma}), \mathbf{f}(\boldsymbol{\varsigma})) d\Omega_{\varsigma}. \tag{15}$$

Expressions (6), (14), (15) are the solution of equations (1), (2) and satisfy relation (11). □

Formulas (6), (14), (15) in the component terms have the form

$$u_{1j}(\mathbf{x}) = \frac{\rho}{4\pi\nu_1} \int_{\Omega} \frac{f_j(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_1} \sum_{m=1}^{3} \int_{\Omega} f_m(\boldsymbol{\varsigma}) \frac{\partial^2 R(\mathbf{x}, \boldsymbol{\varsigma})}{\partial x_m \partial x_j} d\Omega_{\varsigma}, \tag{16}$$

$$u_{2j}(\mathbf{x}) = \frac{\rho}{4\pi\nu_2} \int_{\Omega} \frac{f_j(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_2} \sum_{m=1}^{3} \int_{\Omega} f_m(\boldsymbol{\varsigma}) \frac{\partial^2 R(\mathbf{x}, \boldsymbol{\varsigma})}{\partial x_m \partial x_j} d\Omega_{\varsigma}, \tag{17}$$

$$p(\mathbf{x}) = -\frac{\rho}{4\pi} \mathrm{div} \int_{\Omega} \frac{\mathbf{f}(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} - \frac{\rho}{4\pi} \sum_{m=1}^{3} \frac{\partial}{\partial x_m} \int_{\Omega} \frac{f_m(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma}. \tag{18}$$

Here $f_m(\boldsymbol{\varsigma})$ is the m-th component of the mass forces vector $\mathbf{f}(\boldsymbol{\varsigma})$.

Consider a particular case, namely, the action of a single force applied to the origin and directed to the axis Ox_i. This force can be interpreted as distribution of the mass forces, determined by the following formula [6]:

$$\mathbf{f}(\mathbf{x}) = \mathbf{E}\,\delta(\mathbf{x}). \tag{19}$$

Here $\mathbf{E}$ is the square matrix of dimension 3 with elements δ_{ij} $(i, j = 1, 2, 3)$, δ_{ij} is the Kronecker delta, $\delta(\mathbf{x})$ is the Dirac function of a vector argument. Substituting (19) into (14) and (15) we obtain

$$u_{ij}^{k}(\mathbf{x}) = \frac{\rho\,\delta_{ij}}{4\pi\nu_k} \int_{\Omega} \frac{\delta(\boldsymbol{\varsigma})}{R(\mathbf{x}, \boldsymbol{\varsigma})} d\Omega_{\varsigma} + \frac{\rho}{8\pi\nu_k} \sum_{m=1}^{3} \delta_{im} \int_{\Omega} \delta(\boldsymbol{\varsigma}) \frac{\partial^2 R(\mathbf{x}, \boldsymbol{\varsigma})}{\partial x_m \partial x_j} d\Omega_{\varsigma} \tag{20}$$

$$= \frac{\rho}{4\pi\nu_k}\frac{\delta_{ij}}{R(\mathbf{x},0)} + \frac{\rho}{8\pi\nu_k}\frac{\partial^2 R(\mathbf{x},0)}{\partial x_i \partial x_j},$$
$$(k = 1, 2;\ i, j = 1, 2, 3),$$

and, similarly, by substituting (17) into (16), we obtain:

$$p^i(\mathbf{x}) = -\frac{\rho}{4\pi}\frac{\partial R^{-1}(\mathbf{x},0)}{\partial x_i}, \quad (i = 1, 2, 3). \tag{21}$$

Here $u^i_{kj}(\mathbf{x})$ is the j-th component of the velocity vector $u^i_k(\mathbf{x})$; $u^i_k(\mathbf{x})$ $(k = 1, 2)$ and $p^i(\mathbf{x})$ are, respectively, the velocity and the pressure caused by the action of a single concentrated force directed along the axis Ox_i and applied to the origin. In general, when a concentrated force is applied to an arbitrary point ξ, expressions (20), (21) take the form:

$$u^k_{ij}(\mathbf{x}) = \frac{\rho}{4\pi\nu_k}\frac{\delta_{ij}}{R(\mathbf{x},\xi)} - \frac{\rho}{8\pi\nu_k}\frac{\partial^2 R(\mathbf{x},\xi)}{\partial x_i \partial x_j}, \tag{22}$$
$$(k = 1, 2;\ i, j = 1, 2, 3),$$

$$p^i(\mathbf{x}) = -\frac{\rho}{4\pi}\frac{\partial}{\partial x_i}\frac{1}{R(\mathbf{x},\xi)}, \quad (i = 1, 2, 3). \tag{23}$$

4 Conclusion

In this paper, the solution to describe the three-dimensional stationary flows of viscous fluids of the two-velocity continuum with the pressure phase equilibrium has been constructed. The influence of the physical phase densities, volume saturation and viscosity of substances constituting a two-phase continuum on the flow velocity and pressure has been revealed.

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